

**OPTIMUM CENTRALIZED PORTFOLIO
CONSTRUCTION WITH DECENTRALIZED PORTFOLIO
MANAGEMENT**

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Many financial institutions employ outside portfolio managers to manage part or all of their investable assets. These institutions include pension funds, private endowments (e.g., colleges and charities), and private trusts. Pension funds are the largest and most likely organizations to employ several outside managers, each of whom manages a part of the overall portfolio. In this paper we will use the pension fund manager as the prototype of the centralized decision-maker trying to optimally manage a set of decentralized decision-makers but the analysis is general.

If the centralized decision-maker (CDM) is a mean variance maximizer, the CDM could construct a portfolio using standard portfolio theory and estimates of mean return, variances, and covariances between the portfolios constructed by a group of decentralized managers. However, this overall portfolio is unlikely to be optimum since the individually managed portfolios themselves were constructed without taking into account the portfolios of the other managers. The purpose of this article is to set up a structure that leads to the optimum portfolio from the viewpoint of the CDM when there are multiple managers and their portfolios are constructed without reference to each other.

In the first section we will present a more detailed discussion of the problem. We will then solve the problem for one active manager and multiple passive managers. The model is then generalized to multiple active managers. Next, we present solutions under a simplified structure of the return-generating process. Finally, we discuss the complications when short sales are not allowed.

I. Background

In this section we discuss some background material on the pension investment problem and review the relevant literature. The same considerations hold for private

endowments and trusts. Most pension plans are managed by a centralized decision maker at a firm. Most firms have one person who is principally in charge, although the ultimate responsibility rests with a committee, usually the board. This CDM normally employs outside portfolio managers to construct active portfolios. Index funds are generic products and we will assume the centralized decision maker can potentially select one or more of these. The centralized decision maker's task is fourfold: 1) decide how much to invest in each portfolio, 2) give the outside managers instructions that will result in their making optimum allocations from the point of view of the overall plan, 3) design incentive systems so that the managers will behave optimally, and 4) evaluate and select the portfolio managers. In this paper we deal only with the first two of these problems. Throughout the paper, we assume that the portfolio managers will not provide the centralized decision maker with their return forecasts for individual securities, but will provide aggregate information about the portfolios they hold.

Aspects of this problem have previously been addressed by Treynor and Black (1974) and Sharpe (1981). The Treynor Black article discussed the active passive split when the CAPM described the returns on the passive portfolio, short sales are allowed and the single-index model describes the return generating process. Sharpe develops, with one active and one passive manager, the instructions for the active manager that will result in the active manager producing a globally optimal portfolio. He assumes short sales are allowed and the variance covariance matrix is agreed on by all parties. He also solves for the instructions to be given to the managers that results in a global optimal for the case of two managers following exactly the same set of securities where the centralized decision maker believes the best forecast of a securities alpha is a weighted

average of the two managers alphas and where these weights add to one. In solving this problem he maintains the assumption of short sales allowed and agreement on the variance covariance matrix. Sharpe (1981) could not obtain an exact myopic solution for the case of non-overlapping securities. Our analysis differs from Sharpe (1981) in that we generalize to N managers, have no requirement that they hold the same securities and by employing a multi-factor model can arrive at simple rules for forming myopic optimum portfolios and understanding the weight placed on each security in that portfolio.

II. Separation with a single active and multiple passive manager

In this section of the paper we will assume that a centralized decision maker (CDM) exists who hires a single active manager. We will shortly expand the case to several active managers. We will assume the following: 1) the CDM is a mean variance decision maker, 2) the CDM believes a multi-index model describes the return structure for securities and all indexes in the multi-index model are tradable.

The second point requires some clarification. The CDM believes that returns can be described as being generated by a set of indexes (not necessarily orthogonal) that the CDM can take positions in as passive portfolios. For example, this is consistent with a belief that the return on securities is a function of the market return, the return on a portfolio of small stocks, and/or the return on a portfolio of value or growth stocks. The CDM wishes to consider these sources of risk in making the optimum mean variance decision. For expositional reasons we will analyze the CDM's problem with a two index model though the solution easily generalizes to any number of indexes.

A. The CDM's problem

We start by examining the optimum decision the CDM would make if the CDM had all the information that is available to the active managers. As mentioned earlier, we believe the CDM would not be able to obtain risk adjusted return forecasts for individual securities from the active manager, but for the moment we examine the optimum decision as if the CDM has such information. We will also assume that the CDM does not have perfect faith in the return forecasts of the active manager. This implies that the CDM will take positions in the passive portfolios for two reasons, to obtain diversification across securities so that the portfolio is mean variance efficient, and to eliminate some of the lack of reliability in the analyst's estimates.

In order to specify the return generating process, define

1. R_i is the return on stock i
2. R_F is the risk free rate of interest
3. R_A, R_B is the return on index A and index B respectively
4. b_{iA}, b_{iB} is the sensitivity of stock i to indexes A and B
5. $\mathbf{s}_A^2, \mathbf{s}_B^2$ is the variance of the return on indexes A and B
6. \mathbf{s}_{ei}^2 is the residual risk of stock i from the two-index model
7. \mathbf{a}_i is the risk adjusted return on security i
8. e_i is the residual return for security i
9. The superscript D designates that the decision is from the point of view of the CDM.

Then the return generating process is

$$R_i - R_F = \mathbf{a}_i + \mathbf{b}_{iA}(R_A - R_F) + \mathbf{b}_{iB}(R_B - R_F) + e_i \quad (1)$$

Assume that the CDM had access to the excess return forecasts (\mathbf{a}_i) of the active manager. Furthermore, assume the CDM believes that the best estimate of risk-adjusted excess return is an average of the analysts' forecasts and the value that would occur in equilibrium namely zero. Thus, we define the excess risk adjusted return that the CDM would use as $\mathbf{a}_i^D = W\mathbf{a}_i$ where W is set by CDM between 0 and 1.

To solve this problem and assuming short sales, the CDM can use the standard first order conditions. The investments that can be selected are the N individual securities and the two indexes. The first order condition for security i is

$$\bar{R}_i - R_F = Z_i^D \mathbf{s}_i^2 + \sum_{\substack{j=1 \\ j \neq i}}^N Z_j^D \mathbf{s}_{ij} + Z_A \mathbf{s}_{iA} + Z_B \mathbf{s}_{iB} \quad \text{for } i=1, \dots, N+2 \quad (2)$$

Where

1. N is the number of securities entering into the decision making process
2. Z_i^D is a number proportional to the optimal weight which the CDM would place in security i
3. A, B designate passive portfolios.

If the return generating process described in equation (1) is an accurate description of returns and we recognize that the indexes need not be orthogonal, then we can define the variance and covariance between individual securities as

$$\mathbf{s}_i^2 = \mathbf{b}_{iA}^2 \mathbf{s}_A^2 + \mathbf{b}_{iB}^2 \mathbf{s}_B^2 + 2\mathbf{b}_{iA} \mathbf{b}_{iB} \mathbf{s}_{AB} + \mathbf{s}_{ei}^2 \quad \text{for } i=1, \dots, N$$

$$\mathbf{s}_{ij} = \mathbf{b}_{iA} \mathbf{b}_{jA} \mathbf{s}_A^2 + \mathbf{b}_{iB} \mathbf{b}_{jB} \mathbf{s}_B^2 + \mathbf{b}_{iA} \mathbf{b}_{jB} \mathbf{s}_{AB} + \mathbf{b}_{iB} \mathbf{b}_{jA} \mathbf{s}_{AB} \quad \text{for } i=1 \dots N \quad j=1, \dots, N$$

$$i \neq j$$

For the $N+1$ and $N+2$ securities (the indexes), a simpler form exists. For example, for index A the variance is \mathbf{s}_A^2 and the covariance with index B is \mathbf{s}_{AB} and the covariance with individual securities is

$$\mathbf{s}_{iA} = \mathbf{b}_{iA} \mathbf{s}_A^2 + \mathbf{b}_{iB} \mathbf{s}_{AB}$$

Employing these relationships with the first order condition (2), we get for security i

$$\begin{aligned} W\mathbf{a}_i + \mathbf{b}_{iA}(\bar{R}_A - R_F) + \mathbf{b}_{iB}(\bar{R}_B - R_F) = \\ Z_i^D \mathbf{b}_{iA}^2 \mathbf{s}_A^2 + Z_i^D \mathbf{b}_{iB}^2 \mathbf{s}_B^2 + 2Z_i^D \mathbf{b}_{iA} \mathbf{b}_{iB} \mathbf{s}_{AB} + Z_i^D \mathbf{s}_{ei}^2 + \\ \mathbf{b}_{iA} \sum_{\substack{j=1 \\ j \neq 1}}^N Z_j^D \mathbf{b}_{jA} \mathbf{s}_A^2 + \mathbf{b}_{iB} \sum_{\substack{j=1 \\ j \neq 1}}^N Z_j^D \mathbf{b}_{jB} \mathbf{s}_B^2 + \mathbf{b}_{iA} \sum_{\substack{j=1 \\ j \neq 1}}^N Z_j^D \mathbf{b}_{jB} \mathbf{s}_{AB} + \\ \mathbf{b}_{iB} \sum_{\substack{j=1 \\ j \neq 1}}^N Z_j^D \mathbf{b}_{jA} \mathbf{s}_{AB} + Z_A^D \mathbf{b}_{iA} \mathbf{s}_A^2 + Z_A^D \mathbf{b}_{iB} \mathbf{s}_{AB} + Z_B^D \mathbf{b}_{iB} \mathbf{s}_B^2 + Z_B^D \mathbf{b}_{iA} \mathbf{s}_{AB} \end{aligned} \quad (3)$$

and for the indexes

$$\bar{R}_A - R_F = Z_A^D \mathbf{s}_A^2 + \sum_{j=1}^N Z_j^D \mathbf{b}_{jA} \mathbf{s}_A^2 + \sum_{j=1}^N Z_j^D \mathbf{b}_{jB} \mathbf{s}_{AB} + Z_B^D \mathbf{s}_{AB} \quad (4)$$

$$\bar{R}_B - R_F = Z_B^D \mathbf{s}_B^2 + \sum_{j=1}^N Z_j^D \mathbf{b}_{jB} \mathbf{s}_B^2 + \sum_{j=1}^N Z_j^D \mathbf{b}_{jA} \mathbf{s}_{AB} + Z_A^D \mathbf{s}_{AB} \quad (5)$$

Substituting equations (4) and (5) into equation (3) and simplifying, we get

$$Z_i^D = \frac{\mathbf{a}_i^D}{\mathbf{s}_{ei}^2} = \frac{W\mathbf{a}_i}{\mathbf{s}_{ei}^2} \quad (6)$$

The fraction of the funds the CDM would invest in any security if he or she had full information is

$$\frac{Z_i^D}{Z_A^D + Z_B^D + \sum_{j=1}^N Z_j^D}$$

We will solve for Z_A^D and Z_B^D shortly. They will be solved for by analyzing the problem as an allocation of funds across two passive and an optimum active portfolio. However, to solve for the optimum amount in security i we consider the active portfolio denoted by P as a separate portfolio and look at the optimum composition of this portfolio before we allocate across all three portfolios. We can treat the design of P as a separate portfolio because from equation (6), Z_i^D is not a function of Z_A or Z_B .

The amount to invest in security i in the optimal active portfolio from the viewpoint of the CDM is simply

$$\frac{\mathbf{a}_i^D / \mathbf{s}_{ei}^2}{\sum_{j=1}^N \mathbf{a}_j^D / \mathbf{s}_{ej}^2} \quad (7)$$

Recall that $\mathbf{a}_i^D = w\mathbf{a}_i$. With this substitution the optimal composition from the point of view of the CDM equals

$$\frac{\mathbf{a}_i / \mathbf{s}_{ei}^2}{\sum_{j=1}^N \mathbf{a}_j / \mathbf{s}_{ej}^2}.$$

B. Optimum active portfolio

The CDM can ensure that the active manager will hold the optimal active portfolio from the point of view of the CDM simply by instructing the active manager to

rank all stocks by $\mathbf{a}_i / \mathbf{s}_{ei}^2$ and to hold them in that proportion.¹ This simple instruction

ensures that the active manager will turn over to the CDM the same active portfolio that

¹ If the decentralized manager were simply told to form the optimum active portfolio assuming that he could hold the passive portfolio, he would get the same result as following the direction from the central manager.

the CDM would hold if all the security estimates were supplied directly to the CDM. Optimization for the active portfolio is reached without the active manager giving up private information.

Of course the CDM still has the problem of deciding what fraction of funds to place in the active portfolio and each of the passive portfolios.

C. Solving the aggregate allocation problem

Denote the characteristics of the active portfolio by the subscript P . Then from the viewpoint of the CDM, ignoring for the moment any difficulty of getting information, the problem can be formulated and solved using the following first order conditions.

$$\begin{aligned} \mathbf{a}_P^D + \mathbf{b}_{PA}(\bar{R}_A - R_F) + \mathbf{b}_{PB}(\bar{R}_B - R_F) = \\ Z_P^D(\mathbf{b}_{PA}^2 \mathbf{s}_A^2 + \mathbf{b}_{PB}^2 \mathbf{s}_B^2 + 2\mathbf{b}_{PA} \mathbf{b}_{PB} \mathbf{s}_{AB} + \mathbf{s}_{eP}^2) + Z_A^D \mathbf{b}_{PA} \mathbf{s}_A^2 + Z_B^D \mathbf{b}_{PB} \mathbf{s}_B^2 \\ \bar{R}_A - R_F = Z_P^D(\mathbf{b}_{PA} \mathbf{s}_A^2 + \mathbf{b}_{PB} \mathbf{s}_{AB}) + Z_A^D \mathbf{s}_A^2 + Z_B^D \mathbf{s}_{AB} \\ \bar{R}_B - R_F = Z_P^D(\mathbf{b}_{PB} \mathbf{s}_B^2 + \mathbf{b}_{PA} \mathbf{s}_{AB}) + Z_B^D \mathbf{s}_B^2 + Z_A^D \mathbf{s}_{AB} \end{aligned}$$

These are standard first order conditions, and if everything but the Z 's are known the optimum solution can be reached by solving three simultaneous equations or by using any one of a number of standard software packages. To obtain these estimates, the CDM needs to request the active manager's estimate of the alpha for the active portfolio, the residual risk of the active portfolio and the active portfolio sensitivities to the two indexes. These are the types of estimates the active manager should be willing to supply since they are aggregate portfolio values rather than individual security values.² The

² As stated earlier, we are assuming that the CDM and the active manager are employing identical estimates of the \mathbf{b} 's and residual risks but not return characteristics of each security. This could come about naturally if the risk parameters were estimated from the same commercial service (e.g., BARRA or Wilshire). The CDM could either specify that decentralized managers use a particular commercial service or directly supply the risk parameters for the assumption of our model to hold.

CDM needs to estimate the expected return above the riskless rate and risk on the index funds, the covariance between the passive funds, and the amount of weight (W) to put on the active manager's estimates.

We have now presented a set of conditions under which a centralized decision-maker can optimize portfolio composition while employing one active manager. The next problem to solve is the case of several active, decentralized managers.

III. Multiple active managers

The analysis generalizes to multiple active managers whether these managers follow some or all securities in common or follow independent sections of the market.³ For simplicity we will solve for the case of two active managers, but the analysis easily generalizes. Assume that the CDM has different confidence in the forecasts of each manager and believes that all the managers' estimates are too extreme but that the appropriate estimate is some combination of them.⁴ If we designate the weight the CDM puts on the estimate prepared by manager 1 as W_1 and manager 2 as W_2 . Then

$\mathbf{a}_i^D = W_1 \mathbf{a}_{i1} + W_2 \mathbf{a}_{i2}$. Once again it is necessary for the CDM to supply estimates of betas and residual variances to all active managers either directly or by specifying a service such as BARA. Since \mathbf{s}_{ei}^2 is supplied by the CDM to all managers, it is common and

$$\frac{\mathbf{a}_i^D}{\mathbf{s}_{ei}^2} = W_1 \frac{\mathbf{a}_{i1}}{\mathbf{s}_{ei}^2} + W_2 \frac{\mathbf{a}_{i2}}{\mathbf{s}_{ei}^2} \quad (8)$$

³ Sharpe (1981) did not reach an explicit solution in the case where only some securities were in common across active managers.

⁴ Implicit in what follows is if only one manager follows a security, the CDM assumes the best estimate of the second manager's alpha if he/she followed it would be zero.

Earlier we showed that $\frac{\mathbf{a}_i^D}{\mathbf{s}_{ei}^2}$ was proportioned to the optimum amount that the CDM wished to place in security i if all alphas were supplied to the CDM. The issue we addressed in this section is the instruction to give to the individual managers and the correct proportions to invest in each active portfolio so that the CDM, by combining the portfolios of the active managers, ends up with a fraction of the active portfolio proportional to $\frac{\mathbf{a}_i^D}{\mathbf{s}_{ei}^2}$ in security i .

Summing both sides of equation (8) across all securities

$$\sum \frac{\mathbf{a}_i^D}{\mathbf{s}_{ei}^2} = W_1 \sum \frac{\mathbf{a}_{i1}}{\mathbf{s}_{ei}^2} + W_2 \sum \frac{\mathbf{a}_{i2}}{\mathbf{s}_{ei}^2} \quad (9)$$

If the CDM instructs each manager to rank each security, he or she follows by a $\frac{\mathbf{a}_i}{\mathbf{s}_{ei}^2}$ and to place a fraction of money in each security proportional to this ratio we can define the fraction any manager (e.g., manager 1) places in any security as

$$\frac{\frac{\mathbf{a}_{i1}}{\mathbf{s}_{ei}^2}}{\sum_{j=1}^N \frac{\mathbf{a}_{j1}}{\mathbf{s}_{ej}^2}}.$$

We can then derive some of the attributes of the portfolio which manager 1 (or any manager) will hold.

The risk-adjusted excess return on the portfolio held by manager 1 is

$$\mathbf{a}_{P1} = \frac{\sum_j \left[\left(\frac{\mathbf{a}_{j1}}{\mathbf{s}_{ej}^2} \right) \mathbf{a}_{j1} \right]}{\sum_j \frac{\mathbf{a}_{j1}}{\mathbf{s}_{ej}^2}} \quad (10)$$

and the residual risk of this active portfolio is

$$\mathbf{s}_{eP1}^2 = \frac{\sum_j \left[\left(\frac{\mathbf{a}_{j1}}{\mathbf{s}_{ej}^2} \right)^2 \mathbf{s}_{ej}^2 \right]}{\left[\sum_j \left(\frac{\mathbf{a}_{j1}}{\mathbf{s}_{ej}^2} \right) \right]^2} \quad (11)$$

Taking the ratio of (10) and (11) yields

$$\frac{\mathbf{a}_{P1}}{\mathbf{s}_{eP1}^2} = \sum_j \frac{\mathbf{a}_{j1}}{\mathbf{s}_{ej}^2} \quad (12)$$

Furthermore, \mathbf{b}_{P1K} where the subscript K is a counter, indication either index A or index B equals

$$\mathbf{b}_{P1K} = \frac{\sum_j \left(\frac{\mathbf{a}_j}{\mathbf{s}_{ej}^2} \mathbf{b}_{jK} \right)}{\sum_j \frac{\mathbf{a}_j}{\mathbf{s}_{ej}^2}} \quad (13)$$

Rearranging and substituting equation (12) yields

$$\sum_j \frac{a_{1j} b_{jK}}{s_{ej}^2} = b_{P1K} \frac{a_{P1}}{s_{eP}^2} \quad (14)$$

Having developed these expressions, we can now show that there exists an allocation across the active portfolios along with the instruction to the individual managers to hold stocks in proportion $\frac{a_i}{s_{ei}^2}$, which results in an overall optimum to the CDM.

Substituting equation (12) into (9) yields

$$\frac{a_P^D}{s_{ePD}^2} = W_1 \frac{a_{P1}}{s_{eP1}^2} + W_2 \frac{a_{P2}}{s_{eP2}^2} \quad (15)$$

Recall that the individual portfolio manager has been instructed to form a portfolio by holding securities proportional to the ratio of excess return to residual risk. Recognizing this instruction and using equation (12) to simplify the denominator

$$X_{i1} = \frac{\frac{a_{i1}}{s_{ei}^2}}{\frac{a_{P1}}{s_{eP1}^2}} \quad (16)$$

Utilizing equation (8) and dividing both sides of equation (8) by $\frac{a_P^D}{s_{ePD}^2}$, the correct

amount in security i in the active portfolios from the point of view of the CDM is

$$X_i^D = \frac{\frac{a_i^D}{s_{ei}^2}}{\frac{a_P^D}{s_{ePD}^2}} = W_1 \left[\frac{\frac{a_{i1}}{s_{ei}^2}}{\frac{a_{P1}^D}{s_{ePD}^2}} \right] + W_2 \left[\frac{\frac{a_{i2}}{s_{ei}^2}}{\frac{a_{P2}^D}{s_{ePD}^2}} \right] \quad (17)$$

Since $\frac{\mathbf{a}_P^D}{\mathbf{s}_{ePD}^2}$ can be computed from equation (15), if the CDM obtains \mathbf{a}_{P1} and

\mathbf{s}_{eP1}^2 from manager 1, and \mathbf{a}_{P2} and \mathbf{s}_{eP2}^2 from manager 2, he or she can determine

optimum proportions among active managers. The centralized decision maker simply holds a weighted average of each active manager's portfolio where the weights are

$$\left[\frac{\mathbf{a}_{P1} / \mathbf{s}_{eP1}^2}{W_1 \frac{\mathbf{a}_{P1}}{\mathbf{s}_{eP1}^2} + W_2 \frac{\mathbf{a}_{P2}}{\mathbf{s}_{eP2}^2}} \right] W_1 \text{ and } \left[\frac{\mathbf{a}_{P2} / \mathbf{s}_{eP2}^2}{W_1 \frac{\mathbf{a}_{P1}}{\mathbf{s}_{eP1}^2} + W_2 \frac{\mathbf{a}_{P2}}{\mathbf{s}_{eP2}^2}} \right] W_2 \text{ respectively.}$$

In addition, the CDM can determine the split between the aggregate active portfolio and the passive portfolios using any standard portfolio algorithm by solving the portfolio problem using the \mathbf{a} , \mathbf{b}_{PA} , \mathbf{b}_{PB} , and \mathbf{a}_{eP}^2 from each manager and estimating the weights, the excess return, risk of the passive funds, and the covariance between passive funds centrally. This is done using the overall active portfolio and two passive portfolios using the equations in Section II C.

IV. Orthogonal Indexes

Up to this point we have assumed that the indexes are not orthogonal. The advantage of this is that it allows the passive portfolios to be portfolios that exist in the market such as small stocks, the S&P Index, growth stocks, etc. However, if we are willing to assume orthogonal indexes the allocation across active and passive managers is

simplified. With orthogonal indexes, the covariance among indexes is zero, and the first order condition for the passive index is simpler. For passive index A equation (4) becomes

$$\bar{R}_A - R_F = \sum_j Z_j^D \mathbf{b}_{jA} \mathbf{s}_A^2 + Z_A^D \mathbf{s}_A^2$$

Solving for Z_A^D

$$Z_A^D = \frac{\bar{R}_A - R_F}{\mathbf{s}_A^2} - \sum_{j=1}^N Z_j^D \mathbf{b}_{jA}$$

Substituting for Z_j^D from equation (6) yields

$$Z_A^D = \frac{\bar{R}_A - R_F}{\mathbf{s}_A^2} - \sum_{j=1}^N \frac{\mathbf{a}_j^D}{\mathbf{s}_{ej}^2} \mathbf{b}_{jA}$$

Expressing $\frac{\mathbf{a}_j^D}{\mathbf{s}_{ei}^2}$ in terms of the two active portfolios

$$Z_A^D = \frac{\bar{R}_A - R_F}{\mathbf{s}_A^2} - W_1 \sum_{j=1}^N \frac{\mathbf{a}_{j1}}{\mathbf{s}_{ej}^2} \mathbf{b}_{jA} - W_2 \sum_{j=1}^N \frac{\mathbf{a}_{j2}}{\mathbf{s}_{ej}^2} \mathbf{b}_{jA}$$

Finally, using equation (10):

$$Z_A^D = \frac{\bar{R}_A - R_F}{\mathbf{s}_A^2} - \left[W_1 \frac{\mathbf{a}_{P1}}{\mathbf{s}_{eP1}^2} \mathbf{b}_{P1A} + W_2 \frac{\mathbf{a}_{P2}}{\mathbf{s}_{eP2}^2} \mathbf{b}_{P2A} \right]$$

Thus, the centralized decision maker can determine the total Z and the split between each of the passive portfolios and each of the active portfolios using a simple formula if all managers provide their estimates of \mathbf{a}_P , \mathbf{s}_{eP}^2 and \mathbf{b} on each index, and the centralized

decision maker estimates the W 's and excess return and risk on the index. The active managers also need to have common risk measures, \mathbf{b}_i 's and \mathbf{s}_{ei}^2 for all securities under consideration.

V. Short sales not allowed

The final issue to examine is whether a rule can be worked out if short sales are not allowed. The answer is yes, but only under very restrictive conditions. From the point of view of the centralized decision maker, if there is a single active manager, then

the Z_i in a particular security is $Z_i^D = \frac{w\mathbf{a}_i}{\mathbf{s}_{ei}^2}$ if all the passive portfolios are held long.

Using the analysis presented earlier along with Kuhn Tucker conditions, an explicit solution exists. The simple form for Z_i^D allows a construction of a simple rule for the active manager. Select all securities with positive \mathbf{a}_i , and the previously presented analysis follows.

What happens if one or more of the passive portfolios is short-sold in an optimum solution? If the CDM allows a shorting of passive portfolios, possibly by using futures, but forbids short sales of securities in the active portfolio, then a single ranking by

$\frac{\mathbf{a}_i}{\mathbf{s}_{ei}^2}$ is optimum where all stocks with positive alphas are held in proportion to this

ratio. Finally, if all but one index is held long, repetitive substitution will result in a set of first order conditions that are solvable as in the simple rules of Elton, Gruber & Padberg (1976).

If there are multiple active managers, even more stringent assumptions are necessary to obtain an optimum solution. To understand the problem, consider the case where manager 1 forecasts $\mathbf{a}_{i1} > 0$ and manager 2 forecasts $\mathbf{a}_{i2} < 0$ where the absolute value of \mathbf{a}_{i2} is greater than \mathbf{a}_{i1} and the CDM puts equal weights on the estimates of each manager. In this case the CDM would want to hold zero in security i . However, manager 1 will hold positive proportions and without short sales, manager 2 will hold zero rather than short sell. No combination will provide an optimum to the CDM.

The only exception to this scenario is the case where the centralized manager wishes to place no weight on a forecast of a negative alpha. This implies that the CDM believes the managers have no ability to forecast below normal returns but have some ability on the upside. In the case where $\mathbf{a}_{i1} > 0$ and $\mathbf{a}_{i2} < 0$, the CDM would want to use $\mathbf{a}_i^D = W_1 \mathbf{a}_{i1}$ and, providing all passive portfolios are held long or short sales of passive portfolios are allowed, the analysis outlined above goes through with each active manager not allowed to have short sales.

VI. Conclusion

In this article we have shown that under realistic conditions when short sales are allowed, it is possible, and indeed quite easy, for a centralized decision maker to form an optimal overall portfolio while employing multiple outside portfolio managers. Outside managers should be willing to supply the information the CDM needs since it does not require them to reveal private information on individual securities. Managers should be hesitant to reveal information on individual securities, since it is useful for multiple portfolios and to reveal it opens up the possibility of resale of the information.

Unfortunately, a general solution does not exist when short sales are not allowed. This is a problem ignored in the past literature. While we cannot present a general optimum model when short sales are not allowed, we have pointed out conditions under which the models we developed for decentralized management holds when short sales are not allowed. If there is a single active manager to combine with passive indexes, a solution exists if it is optimum for the manager to place some funds in each index and/or the indexes (as opposed to the securities) can be sold short. If the indexes cannot be sold short, a solution still exists as long as one and only one index is not held long.

In the case of multiple active managers, the analysis in the previous paragraph holds as long as a forecast of a negative alpha by a manager is taken to convey no information and the manager is simply told not to hold securities with negative alpha.

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